Antiferromagnetic triangular Ising model: an exact calculation of P(h)

1983 J. Phys. A: Math. Gen. 16 L265

(http://iopscience.iop.org/0305-4470/16/8/002)

J. Phys. A: Math. Gen. 16 (1983) L265–L268. Printed in Great Britain

LETTER TO THE EDITOR

Antiferromagnetic triangular Ising model: an exact calculation of $P(h)$

T C Choy and D Sherrington
Physics Department, Imperial College, London, SW7 2BZ, UK

Received 23 February 1983

Abstract. Using exact results for the nearest-neighbour two, four and six spin correlations, we have evaluated the local field probability distribution function $P(h)$ for the (fully frustrated) triangular antiferromagnetic Ising lattice in its ground state.

In this communication we present an exact evaluation of $P(h)$, the distribution of local molecular fields h, for a nearest-neighbour Ising antiferromagnet on a triangular net, a periodically frustrated (Toulouse 1977) system which does not order at any temperature (Wannier 1950, Houtappel 1950) in the absence of perturbations, but which can be made to order by a suitable small change of exchange parameters (Newell 1950) or by the application of a field (Schick *et al* 1977) or, apparently, by dilution (Grest and Gabl 1979).

The motivation for the investigation stems from observations of spin-glass studies. Computer experiments on spin-glass models with competing exchange sign have universally demonstrated a zero-field minimum in the distribution of local molecular fields in their ground or low-lying metastable states (see e.g. Sherrington 1975, reporting work with Kirkpatrick (unpublished elsewhere), Walker and Walstedt 1977, 1980, Binder 1977, Palmer and Pond 1979, Bantilan and Palmer 1981). A similar minimum has been found in simulations of amorphous antiferromagnets (Khanna and Sherrington 1980, McLenaghan and Sherrington 1983). All these systems exhibit frustration and disorder.

Among periodic systems those with no frustration have $P(0)=0$ in the ground state while those with no interactions have $P(0)=1$. Provided they are above the lower critical dimension for stability against fluctuations, the former of these order, the latter do not. It seems interesting, therefore, to evaluate the $P(h)$ distribution for a system which is intermediate, yet soluble, particularly one known to lie just on the border between ordering and non-ordering. We find $P(h)$ for the triangular Ising model to be almost flat around $h=0$ with a slight zero-field maximum.

Our method of solution makes use of two, four and six spin correlation functions first obtained by Stephenson (1964). For orientation we first present a brief review of the method of solution for these quantities.

While there exist numerous rederivations of Onsager's (1944) solution for the two-dimensional Ising model, the version that is most fruitful for the investigation of spin correlations is due to Kasteleyn (1963), generalised to the triangular lattice by Stephenson (1964); it employs Pfaffians. Since we shall need these results in our calculation, a quick review seems appropriate.

Considering the case of equal bonds, the partition function for a nearest-neighbour Ising model on a lattice of N spins is given by

$$Z_N = \cosh^{3N} K \sum_{\{\sigma\}} \prod_{\langle ij \rangle} (1 + v\sigma_i\sigma_j) \tag{1}$$

where $v = \tanh K$, $K = J/kT$, the summation is over all spin configurations, while the product is over nearest-neighbour spins. In this form the problem is naturally cast as the counting of dimer configurations on a lattice (see Kac and Ward 1952, Potts 1952):

$$Z_N = 2^N \cosh^{3N} K \left(1 + \sum_{r,s,t} v^{r+s+t} p(r, s, t)\right) \tag{2}$$

where $p(r, s, t)$ is the number of polygons with an even number of lines at each vertex that can be constructed with r horizontal, s vertical and t diagonal nearest-neighbour links for a triangular lattice; it is convenient to use a square representation of the triangular lattice as shown in figure 1, with equal interactions horizontally, vertically and along the exhibited diagonals. The final answer is given by

$$Z_N = 2^N \cosh^{3N} K \,\mathrm{Pf}\, \hat{A} \tag{3}$$

where $\hat{A}$ is an antisymmetric matrix whose determinant is the square of the Pfaffian. This result is obtained only after the use of a topological theorem first proved by Kasteleyn which now bears his name (see Stephenson 1964). By appropriate use of some identities for the Ising spins, the calculation of spin correlations proceeds along similar lines after casting the problem as a perturbed partition function with a different weight $1/v$ at the relevant sites.

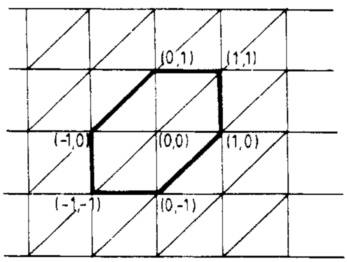

Figure 1. Nearest-neighbour spin correlations on the triangular Ising model.

The result for all even-order correlations can also be expressed as Pfaffians of appropriate antisymmetric matrices. We shall leave the details at this stage but merely quote the results which we shall use. The reader can pursue this from Stephenson (1964) or Montroll *et al* (1963). Labelling the spins about the origin as in figure 1, the following correlations, which are all expressed as Pfaffians with elements given by elliptic integrals, can be evaluated in closed form for the ground state in the interesting case $J < 0$:

$S_{01} = \langle \sigma_{0,1}\sigma_{1,1} \rangle = -\frac{1}{3}$, $\quad S_{14} = \langle \sigma_{-1,-1}\sigma_{1,1} \rangle = (\frac{1}{9} - 3/\pi^2)$,

$S_{13} = \langle \sigma_{-1,0}\sigma_{1,1} \rangle = (\frac{1}{9} + 2\sqrt{3}/\pi)$,

$S_{1234} = \langle \sigma_{1,0}\sigma_{1,1}\sigma_{0,1}\sigma_{-1,0} \rangle = (-\frac{1}{3} + 4\sqrt{3}/3\pi - 3/\pi^2)$,

$$S_{1235} = \langle \sigma_{1,0}\sigma_{1,1}\sigma_{0,1}\sigma_{-1,-1}\rangle = (-\tfrac{13}{27} + 2\sqrt{3}/3\pi - 3/\pi^2 + 3\sqrt{3}/\pi^3),$$
$$S_{1245} = \langle \sigma_{-1,0}\sigma_{0,1}\sigma_{0,-1}\sigma_{1,0}\rangle = (\tfrac{1}{9} + 4\sqrt{3}/3\pi - 6/\pi^2),$$
$$S_{123456} = \langle \sigma_{-1,0}\sigma_{0,1}\sigma_{1,1}\sigma_{1,0}\sigma_{0,-1}\sigma_{-1,-1}\rangle = (-\tfrac{5}{27} + 4\sqrt{3}/3\pi - 9/\pi^2 + 6\sqrt{3}/\pi^3). \quad (4)$$

The object we want to calculate is

$$P(h) = \left\langle \delta\left(h + \sum_{j=1}^{6} \sigma_j\right)\right\rangle \quad (5)$$

where $h = H/J = 0, 2, 4, 6$ and the sum is over the six-membered ring which is nearest neighbour to $\sigma_{0,0}$. Using an exponential integral representation of the Kronecker delta, this is

$$P(h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{ih\theta}\left\langle \exp\left(i\theta \sum_j \sigma_j\right)\right\rangle$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{ih\theta} \cos^6\theta \left\langle \prod_{j=1}^{6} (1 + i\sigma_j \tan\theta)\right\rangle. \quad (6)$$

Since odd moments vanish by symmetry and the spontaneous magnetisation is zero, we are left with the expansion

$$P(h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{ih\theta} \cos^6\theta \left(1 - \tan^2\theta \sum_{(ij)} \langle \sigma_i\sigma_j\rangle \right.$$
$$\left. + \tan^4\theta \sum_{(ijkl)} \langle \sigma_i\sigma_j\sigma_k\sigma_l\rangle - \tan^6\theta \sum_{(ijklmn)} \langle \sigma_i\sigma_j\sigma_k\sigma_l\sigma_m\sigma_n\rangle\right) \quad (7)$$

where the sums are over distinct two, four and six members of the ring. Finally we have

$$P(h) = (2\pi)^{-1}[I_0(h) - S_\alpha I_2(h) + S_\beta I_4(h) - S_\gamma I_6(h)]$$

where

$$S_\alpha = 6S_{01} + 6S_{13} + 3S_{14}, \qquad S_\beta = 6S_{1234} + 6S_{1235} + 3S_{1245}, \qquad S_\gamma = S_{123456},$$

and I_0, I_2, I_4, I_6 are all elementary integrals. The result after normalisation is given in table 1, with the corresponding results for $T = \infty$ for comparison.

Table 1.

	$P(h)$	
h	Ground state	$T = \infty$
0	0.263 0865	0.312 500
±2	0.234 8927	0.234 375
±4	0.122 7224	0.093 750
±6	0.010 8414	0.015 625

We have evaluated exactly the distribution of local molecular fields in the ground state of a uniform nearest-neighbour Ising antiferromagnet on a triangular lattice. The results are given in table 1; they show a slight maximum at $h = 0$ but with an almost flat distribution around. Since this model is known to be just non-ordering, and in view of the situation for other systems mentioned earlier, it is tempting to suggest that a zero-field minimum in $P(h)$ is indicative of a system with a tendency to order, albeit critical fluctuations might prevent this in a true thermodynamic sense.

Extensions to other fully frustrated systems and to the effect of perturbations on the triangular Ising antiferromagnet are under consideration.

It is a pleasure to thank N Rivier for stimulating discussions in the course of this work. Financial assistance from the SERC is gratefully acknowledged.

References

Bantilan F T and Palmer R G 1981 *J. Phys. F: Met. Phys.* **11** 261–6
Binder K 1977 *Z. Phys.* B **26** 339
Grest G S and Gabl E G 1979 *Phys. Rev. Lett.* **43** 1182–5
Houtappel R M F 1950 *Physica* **16** 425–55
Kac M and Ward J C 1952 *Phys. Rev.* **88** 1332
Kasteleyn P W 1963 *J. Math. Phys.* **4** 287
Khanna S N and Sherrington D 1980 *Solid State Comm.* **36** 653
McLenaghan XX and Sherrington D 1983 to be published
Montroll E W, Potts R B and Ward J C 1963 *J. Math. Phys.* **4** 308
Newell G F 1950 *Phys. Rev.* **79** 876–82
Onsager L 1944 *Phys. Rev.* **65** 117
Palmer R G and Pond C M 1979 *J. Phys. F: Met. Phys.* **9** 1451–9
Potts R B 1952 *Phys. Rev.* **88** 352
Schick M, Walker J S and Warlis M 1977 *Phys. Rev.* B **16** 2205–19
Sherrington D 1975 *AIP Conf. Proc.* **29** 224–8
Stephenson J 1964 *J. Math. Phys.* **5** 1009
Toulouse G 1977 *Commun. Phys.* **2** 115–20
Walker L R and Walstedt R E 1977 *Phys. Rev. Lett.* **38** 514–8
—— 1980 *Phys. Rev.* B **22** 3816
Wannier G H 1950 *Phys. Rev.* **79** 357